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Note

Absorbing Boundary Condition and Budden Turning Point Technique for Electromagnetic Plasma Simulations

An algorithm of single masking in the Maxwell equations for electromagnetic particle simulation is discussed. This absorbing boundary condition is analyzed in the light of the Budden turning point condition.

1. Introduction

Since the spatial extent of plasma particle simulation is necessarily finite and bounded, it is quite important to develop techniques which approximately handle the boundaries or extrapolate the system. The simplest to implement is the periodic boundary condition. Most of the basic particle codes have been implemented this way. For electrostatic codes different and more sophisticated boundary condition techniques have been developed. One is the so-called capacity matrix technique [1, 2] which adjusts the boundary condition by adding the boundary charge in real space. This technique can be extended to magnetostatic codes. Another [3] matches the vacuum solutions. Yet another [4] constructs a solution from a sum of the homogeneous solution and the inhomogeneous solution to Poisson's equation. An extension of this to magnetostatic codes is also possible.

On the other hand, the electromagnetic boundary treatment is less developed. Among the most important conditions beside the periodic one is the absorbing boundary condition. One technique [5] originally by Dawson and Langdon utilizes the fact that the hyperbolic operator $\partial_t^2 - c^2\,\partial_x^2$ in the Maxwell equation can be factorized into the left- and right-ward propagation operators $\partial_t \pm c\partial_x$. This technique can be employed only in strictly one dimension. To date the most complete method for this problem has been developed by Lindman [6]. Lindman's method utilizes the projection operators (left- and right-ward), which allow oblique angle incidence to be taken, and approximates the operators by a numerically stable partial fraction expression (a Padé approximation). The technique requires solving several (three or six) finite difference equations (in order to restrict the amplitude error to less than 1%) to determine the "reflection" coefficients. This procedure has to be updated in time. The method has been successfully implemented including in ZOHAR; however, it presently handles only one directional absorption [7].

A naive approach by extrapolation cannot yield a complete absorption. Consider a three-point extrapolation, for example,

$$E_{-1} - 3E_0 + 3E_1 - E_2 = 0, \tag{1}$$

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where subscripts indicate the grid position, i.e., 0 means on the boundary and -1 the extrapolated grid and so on. Let us express the solution at the nth grid as

$$E_n = Ae^{inb} + Be^{-inb}, \tag{2}$$

where the coefficient A indicates the amplitude of the outgoing wave and B the incoming one. Substituting Eq. (2) into Eq. (1) yields

$$B = Ae^{-2ib}\left(\frac{1-e^{ib}}{1-e^{-ib}}\right)^3. \tag{3}$$

It follows from Eq. (3) that $|B| = |A|$. We, therefore, conclude that the extrapolation cannot make the incoming weve amplitude vanish ($|B| = 0$ or $|B| \ll |A|$) and that no absorbing condition is achieved. Equation (3), however, suggests that extrapolation with damping (complex b) may lead to $|B| \ll |A|$, but not $|B| = 0$. (A more sophisticated boundary treatment for absorption or open boundaries by extrapolation is discussed by Orlanski [8].) Another simple approach to absorbing boundary conditions is the method of coordinate stretching [9, 10]. Beyond the physical volume where the grid is regular, the grid spacing in the ramp is exponentially stretched so that over 10 grid points, for example, the actual distance is $\sum_{i=1}^{10} a^i \Delta_x$, where a is a ratio of stretching from one grid point to the next neighbor and Δ_x the regular grid distance. If this distance is long enough, the information traveling over the distance of the ramp might never come back in a practical simulation time span even if the wave is reflected at the last edge point. This method, however, also suffers from incomplete absorption. The finite difference of the distance in grid space is $x(j+1) - x(j) = a^j \Delta_x$ in the ramp. Consider the wave equation $\phi'' + k^2\phi = 0$:

$$(D_x^2 + k^2)\phi = [\partial_x^2 + p\,\partial_x + k^2]\phi = 0, \tag{4}$$

where D_x is the finite difference operator and $p = 2(1-a)/(1+a)\Delta_x$. For $p^2 < 4k^2$, the stretching adds some effective imaginary part to the wavenumber k. On the other hand, too much stretching $p^2 > 4k^2$ severely distorts the nature of the wave equation, leading to complete reflection. We have thus seen that naive or simple techniques considered above cannot lead to the completely absorbing boundary condition. In this article we present an algorithm of single masking in the Maxwell equations which provides a good absorbing boundary condition for electromagnetic waves in electromagnetic particle codes.

2. Single Masking Algorithm

Let us introduce a masking function $f(x)$. A masking procedure of an electric field, for example, is simply

$$\tilde{E}(x) = f(x)\,E(x), \tag{5}$$

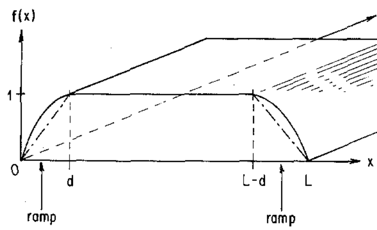

FIG. 1. Masking function $f(x)$. A slight variation, a linear ramp, is also shown in broken lines.

where we consider a one dimensional case for simplicity. We adopt the masking function as parabolically smoothly matched at the plasma (or real system) boundary:

$$\begin{aligned} f(x) &= -d^{-2}x^2 + 2d^{-1}x && (0 \leqslant x < d) \\ &= 1 && (d \leqslant x \leqslant L-d) \\ &= -d^{-2}x^2 + 2d^{-2}(L-d)x + 1 + d^{-2}(L-d)^2 && (L-d < x \leqslant L), \end{aligned} \tag{6}$$

where d is the length of the ramps and L the total system length (see Fig. 1). Note that the way $f(x)$ approaches $x = x_0$ (x_0: 0 or L) is linear: $f(x) \propto (x - x_0)$. The plasma is contained in $d \leqslant x \leqslant L - d$. We mask either transverse electric fields E_y and E_z or magnetic fields B_y and B_z, but not both. We call this a single masking procedure. For each time step, the current (and charge) generates electromagnetic fields. We then mask, say, the transverse electric fields by Eq. (5). When we update the transverse electric fields, the old values assume these masked fields. The algorithm to calculate the Maxwell equations is otherwise exactly the same as the conventional code. Since particles are confined in $d \leqslant x \leqslant L - d$, either masked or unmasked fields make no difference to the fields which push particles in $d \leqslant x \leqslant L - d$. If the system is two dimensional, the generalization of the masking procedure is straightforward: when the absorbing ramps are located parallel to the y axis from $x = 0$ to $x = d$ and from $x = L - d$ to $x = L$, the masking is done for the transverse electric fields (or magnetic fields) whose directions are perpendicular to the normal to the boundary. For three dimensions in cartesian coordinates, we have not found a generalization of the present masking procedure. We tested this algorithm in both one and two dimensions with both one and two dimensional absorbing boundaries; the code is numerically stable and produces physically desired results, namely, absorption of outgoing waves without any unphysical side effects. When we make both the transverse electric fields and magnetic fields (double masking), however, the code is numerically unstable: it amplifies a signal entering the masked region.

Figure 2 shows an electromagnetic pulse launched in a plasma to the left. The pulse is such as is described in [11] with self-consistent velocity modulation. In this test the wave amplitude is very large (relativistic): $eE/(m\omega c) \simeq eB/(m\omega c) \simeq 1$, where m and e are the electron mass and charge, ω is the wave frequency, and c is the speed of light. We took $c = 9\omega_{\mathrm{p}}\Delta$, the size of the ramps $d = 10\Delta$, and the total system

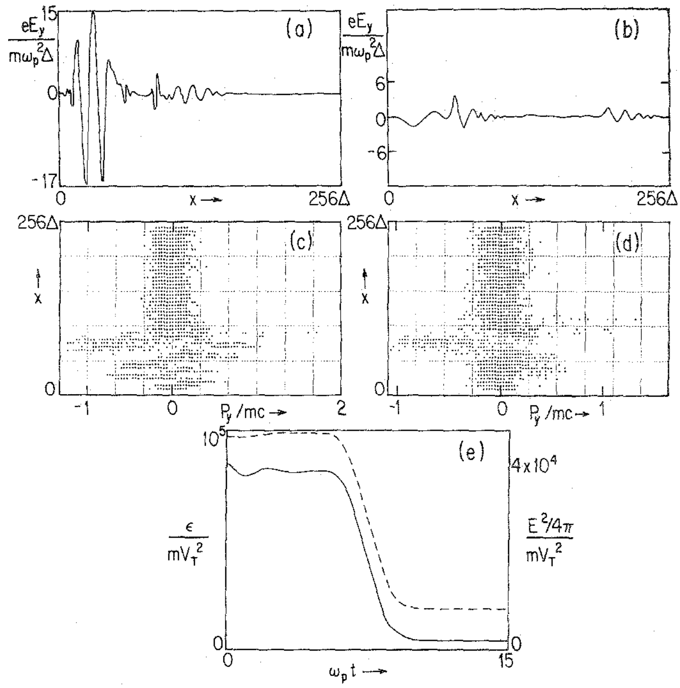

FIG. 2. Large amplitude electromagnetic pulse propagation into the boundary. (a) The transverse electric field E_y vs x at $t = 4\omega_p^{-1}$. The pulse comes into the boundary $x = 0$ from right to left. (b) E_y vs x at $t = 14\omega_p^{-1}$. The pulse has gone through $x = 0$. The relative scale of E_y is the same as in (a). (c) The phase space of electrons (p_y, x) at $t = 5\omega_p^{-1}$ before absorption. (d) The phase space (p_y, x) at $t = 15\omega_p^{-1}$ after absorption. (e) The wave electromagnetic energy (real line) and total energy (broken line) in time.

length $L = 256\Delta$, where ω_p is the plasma frequency and Δ the unit grid length. The wave is a linearly polarized (E_y and B_z) electromagnetic pulse of length $l = \pi c/\omega_p$ and wavenumber $k = 2\pi/15\Delta$. The transverse electric fields [shown in Figs. 2a and b] as well as the magnetic fields are absorbed. When the pulse goes into the ramp, the field amplitude decreases and creates little reflection or transmission. Figure 2e shows the wave energy and kinetic energy plotted in time and indicates that these energies decrease as the wave is being absorbed and stay constant after the pulse is absorbed. The wave energy of the residual noise after absorption is about 4% of the original in this case. The case without a plasma with $d = 10\Delta$ observed 3% of residual noise after transmission of the wavepacket through the ramp. With $d = 5\Delta$, 20%; while with $d = 20\Delta$, 0.4%. When we used a triangular ramp with $d = 10\Delta$ without a plasma, slightly more (4%) residual noise was observed. Two mechanisms can be responsible for the residual noise. First, our theory below deals with a continuum, while the code calculation is done on a discrete grid. Second, long wavelength ($\lambda > d$) Fourier components of the pulse do not see the linear dependence $f(x) \propto x - x_0$ of the masking function in the neighborhood of $x = x_0$. Figure 3 shows two dimensional

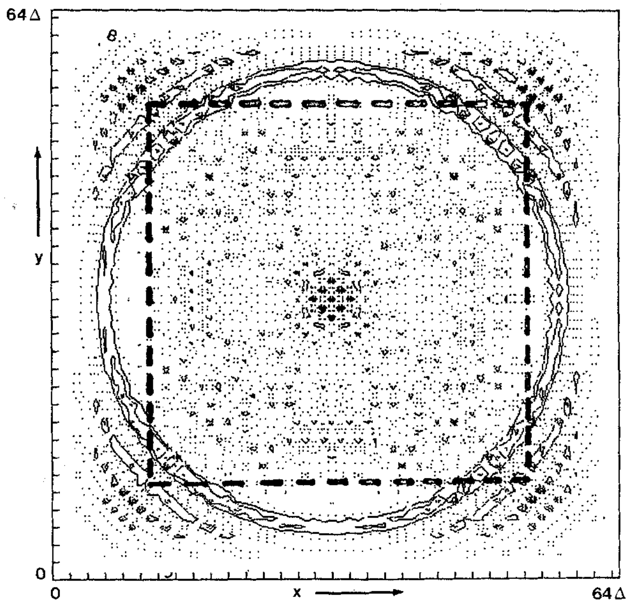

FIG. 3. Two dimensional wave front. The response of E_z to a point source $E_z(t=0)$ located at the middle is shown by contours of equal amplitudes of E_z. The system is $L_x \times L_y = 64\Delta \times 64\Delta$ with the beginning of ramps indicated by broken lines. Dotted lines indicate negative values.

propagation of a wave front from a point source located in the middle of the x–y plane. The case contains no plasma. The ramps are two dimensional. One can see that the wave front approaching a ramp decreases in amplitude and eventually vanishes at either $x=0$ (and L_x) or $y=0$ (and L_y).

3. The Budden Turning Point and Complete Absorption

The singly masked Maxwell equations (electric field masked) in one dimension in the ramp area may be written as

$$\begin{aligned} -i\omega B &= -c\,\partial_x(fE), \\ -i\omega E &= c\,\partial_x B. \end{aligned} \tag{7}$$

If it is two dimensional with the ramps along the x direction, we have

$$\begin{aligned} -i\omega E_x &= ik_y c B_z, \\ -i\omega E_y &= -c\,\partial_x B_z, \\ -i\omega E_z &= -ik_y c B_x + c\,\partial_x B_y, \\ -i\omega B_x &= ik_y c E_z, \end{aligned} \tag{8}$$

$$-i\omega B_y = c\,\partial_x(fE_z),$$
$$-i\omega B_z = ik_y cE_x - c\,\partial_x(fE_y).$$

From Eq. (7) we obtain

$$(\partial_x^2 f + \omega^2/c^2)E = 0. \tag{9}$$

Alternatively we obtain from Eq. (8), from example,

$$\left(\partial_x^2 f + \frac{\omega^2/c^2}{1 - k_y^2/(\omega^2/c^2 - k_y^2)}\right) E_y = 0. \tag{10}$$

If f is linearly proportional to x at $x = 0$ or $x = L$, Eq. (9) [or Eq. (10)] can be written with $\mathscr{E} = fE$ as

$$\left(\partial_x^2 + \frac{\alpha}{x + i\eta}\right)\mathscr{E} = 0, \tag{11}$$

where α is a real number and η is an infinitesimal real number. The differential equation (11) has a singular turning point at $x = 0$. The connection formula of asymptotic solutions for the singular turning point is [12, 13]

$$k_1^{-1/2}\{(A - iB)\exp[i(\xi_1 + \pi/4)] + (A + iB)\exp[-i(\xi_1 + \pi/4)]\}$$
$$\leftrightarrow |k_2|^{-1/2}[(A \pm iB)\exp(|\xi_2|) + 2B\exp(-|\xi_2|)], \tag{12}$$

where subscripts 1 and 2 refer to before and behind the turning point $x = 0$ (or $x = L$). ξ is a normalized coordinate of x. If the wave is coming from left of the turning point, the term proportional to $\exp[i(\xi_1 + \pi/4)]$ is the incoming wave and the other $\exp[-i(\xi_1 + \pi/4)]$ is the reflected outgoing wave. When we demand that the solution be regular at $|\xi_2| \to \infty$, the Budden condition $A - iB = 0$ for $\eta < 0$ is required. As is seen from Eq. (12), the Budden condition leads to a vanishing coefficient of the reflected wave amplitude. Thus we find that the masking function $f(x)$ linearly proportional to $x - x_0$ (x_0 being either end of the system) results in a complete absorption of a plasma wave at $x = x_0$. This is the basic reason for absorption of electromagnetic waves by virtue of the above prescribed algortihm: the masking function (6) has an appropriate form of $(x - x_0)^{-1}$ near $x = x_0$. When we chose f as the one shown by broken lines in Fig. 1, we still observe high absorption but slightly worse than the case with f as given by Eq. (6). This may be due to lack of smooth joining of f at $x = d$ or $x = L - d$. This masking method may be physically looked upon as changing the electric (or magnetic) permeability primarily in a reactive way. Earlier works [14, 15] have resorted to resistive methods, which theoretically cannot absorb 100% of waves over a finite length. On the other hand, the double masking yields $(\partial_x f \partial_x f + \omega^2/c^2)E = 0$ instead of Eq. (9) does not possess a simple Budden turning point.

In conclusion, the simple masking algorithm of the Maxwell equation in one or two dimensions gives a satisfactory result for absorption of electromagnetic waves.

Although the absorption with this method appears worse than that with Ref. [6], the present method is simple, flexible, and more versatile. It is easy to implement and easy to physically understand. The higher dimensional ramps in higher dimensional codes, therefore, can be written with a similar ease in one dimension. If the system is cylindrical and the boundary $r = r_0$ (r: the radial coordinate) is to be the absorbing boundary, it is also possible to apply the present technique to satisfy an approximately absorbing condition as long as the relation (the unit grid length in the radial direction Δ_r) $\ll$ (the boundary r_0) holds. In this case, it is possible to construct a three dimensional code (r, θ, z) with ramps in two directions (r, z).

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T. Tajima
Institute for Fusion Studies
University of Texas
Austin, Texas 78712

Department of Physics
University of California
Los Angeles, California 90024

Y. C. Lee
Department of Physics
University of Maryland
College Park, Maryland 20742